A STUDY OF THE REPRESENTATIONS SUPPORTED BY THE ORBIT CLOSURE OF THE DETERMINANT

SHRAWAN KUMAR

1. Introduction

Let $\mathfrak{v}$ be a complex vector space of dimension m and let $E := \operatorname{End}\mathfrak{v}$. Consider $\mathscr{D} \in Q := S^m(E^*)$, where $\mathscr{D}$ is the function taking determinant of any $X \in \operatorname{End}\mathfrak{v}$. Fix a basis $\{v_1, \dots, v_m\}$ of $\mathfrak{v}$ and a positive integer $n < m$ and consider the function $\mathscr{P} \in Q$, defined by $\mathscr{P}(X) = x_{1,1}^{m-n}\operatorname{perm}(X^o)$, X^o being the component of X in the right down $n \times n$ corner, where any element of $\operatorname{End}\mathfrak{v}$ is represented by a $m \times m$-matrix $X = (x_{i,j})_{1\le i,j,\le m}$ in the basis $\{v_i\}$ and perm denotes the permanent. The group $G = GL(E)$ canonically acts on Q. Let $\mathcal{X}$ (resp. $\mathcal{Y}$) be the G-orbit closure of $\mathscr{D}$ (resp. $\mathscr{P}$) inside Q. Then, $\mathcal{X}$ and $\mathcal{Y}$ are closed (affine) subvarieties of Q which are stable under the standard homothety action of $\mathbb{C}^*$ on Q. Thus, their affine coordinate rings $\mathbb{C}[\mathcal{X}]$ and $\mathbb{C}[\mathcal{Y}]$ are nonnegatively graded G-algebras over the complex numbers $\mathbb{C}$. Clearly, $\mathscr{D} \odot \operatorname{End}E \subset \mathcal{X}$, where $\operatorname{End}E$ acts on Q on the right via: $(q \odot g)(X) = q(g \cdot X)$, for $g \in \operatorname{End}E, q \in Q$ and $X \in E$.

For any positive integer n, let $\bar{m} = \bar{m}(n)$ be the smallest positive integer such that the permanent of any $n \times n$ matrix can be realized as a linear projection of the determinant of a $\bar{m}\times\bar{m}$ matrix. This is equivalent to saying that $\mathscr{P} \in \mathscr{D} \odot \operatorname{End}E$ for the pair $(\bar{m}, n)$. Then, Valiant conjectured that the function $\bar{m}(n)$ grows faster than any polynomial in n (cf. [V]).

Similarly, let $m = m(n)$ be the smallest integer such that $\mathscr{P} \in \mathcal{X}$ (for the pair (m, n)). Clearly, $m(n) \le \bar{m}(n)$. Now, Mulmuley-Sohoni strengthened Valiant's conjecture. They conjectured that, in fact, the function $m(n)$ grows faster than any polynomial in n (cf.[MS1], [MS2] and the references therein). They further conjectured that if $\mathscr{P} \notin \mathcal{X}$, then there exists an irreducible G-module which occurs in $\mathbb{C}[\mathcal{Y}]$ but does not occur in $\mathbb{C}[\mathcal{X}]$. (Of course, if $\mathscr{P} \in \mathcal{X}$, then $\mathbb{C}[\mathcal{Y}]$ is a G-module quotient of $\mathbb{C}[\mathcal{X}]$.) This Geometric Complexity Theory programme initiated by Mulmuley-Sohoni provides a significant mathematical approach to solving the Valiant's conjecture (in fact, strengthened version of Valiant's conjecture proposed by them).

By [K, Theorem 5.2], if an irreducible G-module $V_E(\lambda)$ (with highest weight λ) appears in $\mathbb{C}[\mathcal{Y}]$, then $V_E(\lambda)$ is a polynomial representation of G

given by a partition

$$\lambda : (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n^2+1} \geq 0 \geq \ldots \geq 0)$$

with last $m^2 - (n^2 + 1)$ zeroes.

From now on (in this Introduction), we assume that m is even. Our principal result in this paper (Corollary 3.2) asserts that for any partition $\mu : (\mu_1 \geq \ldots \geq \mu_m \geq 0 \geq \ldots \geq 0)$ with last $m^2 - m$ zeroes, the irreducible G-module $V_E(m\mu)$ appears in $\mathbb{C}[\mathcal{X}]$ with nonzero multiplicity. In particular, if $m \geq n^2 + 1$, for any irreducible representation $V_E(\lambda)$ appearing in $\mathbb{C}[\mathcal{Y}]$, $V_E(m\lambda)$ appears in $\mathbb{C}[\mathcal{X}]$. Thus, finding an irreducible representation in $\mathbb{C}[\mathcal{Y}]$ which does not occur in $\mathbb{C}[\mathcal{X}]$ (on which the success of the Mulmuley-Sohoni programme relies) for $m \geq n^2 + 1$ is not so easy. As a consequence of our Corollary 3.2, we deduce that the Kronecker coefficient $k^{m\bar{\lambda}}_{d\bar{\delta}_m, d\bar{\delta}_m} > 0$ for any partition $\bar{\lambda} : \left(\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_m \geq 0\right)$ of d, where $\bar{\delta}_m$ is the partition $\bar{\delta}_m : (1 \geq 1 \geq \cdots \geq 1)$ (m factors) (cf. Corollary 3.5).

By a result of Howe (cf. Corollary 2.4), for any fundamental weight ω_i ($1 \leq i \leq m^2 = \dim E$) of $GL(E)$, the irreducible $GL(E)$-module $V_E(d\omega_i)$, for $0 < d < m$, does not occur in $S^\cdot(S^m(E))$, whereas $V_E(m\omega_i)$ occurs with multiplicity one in $S^\cdot(S^m(E))$. In fact, it occurs in $S^i(S^m(E))$. We give an explicit construction of the highest weight vector $P_i = \gamma_{m,i}$ in this unique copy of $V_E(m\omega_i)$ in $S^i(S^m(E))$ (cf. § 2.5).

Our principal Theorem 3.1 asserts that for $1 \leq i \leq m$, P_i does not vanish identically on the orbit $GL(E) \cdot \mathscr{D}$. In particular, $V_E(m\omega_i)$ (for $1 \leq i \leq m$) occurs in $\mathbb{C}[\mathcal{X}]$ with multiplicity one. (As mentioned above, $V_E(d\omega_i)$, for any $0 < d < m$ and $1 \leq i \leq m^2$, does not occur in $S^\cdot(S^m(E))$; in particular, it does not occur in $\mathbb{C}[\mathcal{X}]$.)

To prove our Theorem 3.1, it suffices to show that P_i (for $1 \leq i \leq m$) does not vanish identically on the orbit $\mathscr{D} \odot \mathrm{End}(E)$. To this end, we consider certain special elements $A \in \mathrm{End}(E)$ (as given in the beginning of Section 4). Then, we give an explicit expression for P_i (for any $1 \leq i \leq m^2$) evaluated on the elements $\mathscr{D} \odot A$ (cf. Proposition 4.1). Further specializing A, we show the nonvanishing of $P_i(\mathscr{D} \odot A)$ for $1 \leq i \leq m$ in Section 4.2, where we need to deal with two cases: $1 \leq i \leq m/2$ and $m/2 < i \leq m$ separately.

In Section 5, we show that P_{m^2} vanishes identically on $\mathcal{X}$; in particular, $V_E(m\omega_{m^2})$ does not occur in $\mathbb{C}[\mathcal{X}]$ (cf. Proposition 5.1). We give another expression of $P_i(\mathscr{D} \odot A)$ in Proposition 5.3.

Finally, in Remark 5.5 (b), we observe that $V_E(m\omega_i)$ (for any $1 \leq i \leq m$) occurs in $\mathbb{C}[\overline{GL(E) \cdot \mathcal{P}}]$ with multiplicity one, where $\mathcal{P}$ is the function $E \to \mathbb{C}$ taking any matrix $A \in E := \mathrm{End}\,\mathfrak{v}$ to its permanent. (Of course, as mentioned above, $V_E(d\omega_i)$, for any $0 < d < m$ and $1 \leq i \leq m^2$, does not occur in $S^\cdot(S^m(E))$, and hence it does not occur in $\mathbb{C}[\overline{GL(E) \cdot \mathcal{P}}]$.

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2. An explicit realization of multiples of fundamental $GL(E)$-representations in $S^\cdot(S^\cdot(E))$

Let E be a finite dimensional complex vector space with basis $\{e_1, \dots, e_\ell\}$. Let ω_i, $1 \le i \le \ell$, be the i-th fundamental weight of $GL(E) = GL(\ell)$.

Lemma 2.1. *For any positive integers d, j and m, the multiplicity of the irreducible $GL(E)$-module $V_E(d\omega_i)$ (with highest weight $d\omega_i$) in $S^j(S^m(E))$ is the same as the multiplicity of the irreducible $GL(E_i)$-module $V_{E_i}(d\omega_i)$ in $S^j(S^m(E_i))$, where E_i is the subspace of E spanned by $\{e_i, \dots, e_i\}$.*

In fact, the highest weight vectors in $S^j(S^m(E))$ for the irreducible $GL(E)$-module $V_E(d\omega_i)$ coincide with the highest weight vectors in $S^j(S^m(E_i))$ for the irreducible $GL(E_i)$-module $V_{E_i}(d\omega_i)$.

Proof. Let B_E be the standard Borel subgroup of $GL(E)$ consisting of all the invertible upper triangular matrices (with respect to the basis $\{e_1, \dots, e_\ell\}$). Let $v \in S^j(S^m(E))$ be a B_E-eigenvector of weight $d\omega_i$. Then, clearly $v \in S^j(S^m(E_i))$ and v is a B_{E_i}-eigenvector of weight $d\omega_i$. Conversely, let $v' \in S^j(S^m(E_i))$ be a B_{E_i}-eigenvector of weight $d\omega_i$. Then, the line $\mathbb{C}v'$ is clearly stable under B_E. Moreover, the vector v' is a weight vector of weight $d\omega_i$ with respect to the standard maximal torus T_E (consisting of invertible diagonal matrices) of $GL(E)$. This proves the lemma. ◻

Corollary 2.2. *With the notation as above, the multiplicity $\mu_E(d\omega_i)$ of $V_E(d\omega_i)$ in $S^j(S^m(E))$ is equal to the dimension of the invariant space $[S^j(S^m(E_i))]^{SL(E_i)}$ if $di = jm$. If $di \neq jm$, $\mu_E(d\omega_i) = 0$.*

We recall the following result from [H, Proposition 4.3].

Proposition 2.3. *Let E be a vector space of dimension ℓ as above. For positive integers j, m, we have*

(a) $[S^j(S^m(E))]^{SL(E)} = (0)$, *if* $0 < j < \ell$

(b) $[S^l(S^m(E))]^{SL(E)} \simeq \begin{cases} (0), & \textit{if } m \textit{ is odd} \\ \mathbb{C}, & \textit{if } m \textit{ is even.} \end{cases}$

Combining Corollary 2.2 with Proposition 2.3, together with the action of the center of $GL(E)$, we get the following result.

Corollary 2.4. *Let E be a vector space of dimension ℓ as above. Let m be a positive even integer and let $l \le i \le \ell$. Let d be the smallest positive integer such that $V_E(d\omega_i)$ occurs in $S^\cdot(S^m(E))$ as a $GL(E)$-submodule. Then, $d =$*

m. Moreover, $V_E(m\omega_i)$ occurs in $S^{\bullet}(S^m(E))$ with multiplicity 1 and it occurs precisely in $S^i(S^m(E))$.

From now on, until further notice, m is an even positive integer.

We first give an explicit construction of the invariant $[S^i(S^m(E_i))]^{SL(E_i)}$ for any $1 \le i \le \ell$. Recall from Proposition 2.3 that it is one dimensional.

2.5. **An explicit construction of $[S^i(S^m(E_i))]^{SL(E_i)}$.** Recall that E_i has a basis $\{e_1, \dots, e_i\}$. Let $M(i,i)$ be the space of $i \times i$ matrices over $\mathbb{C}$. Define a linear isomorphism

$$\theta : \left(\otimes^2 E_i\right)^* \xrightarrow{\sim} M(i,i), \quad \theta(f) = \left(\theta(f)_{p,q}\right)_{1 \le p,q \le i},$$

where $\theta(f)_{p,q} = f\left(e_p \otimes e_q\right)$, for any $f \in \left(\otimes^2 E_i\right)^*$.

Let $GL(E_i)$ act on $M(i,i)$ via

$$g \cdot A = \left(g^{-1}\right)^t A g^{-1}, \ \text{ for } g \in GL(E_i) \text{ and } A \in M(i,i).$$

Then, θ is $GL(E_i)$-equivariant. Now, define the map

$$\theta^{\otimes m/2} : (\otimes^m E_i)^* \xrightarrow{\sim} \otimes^{m/2}(M(i,i))$$

by identifying

$$(\otimes^m E_i)^* \simeq \left(\left(\otimes^2 E_i\right)^*\right) \otimes \cdots \otimes \left(\left(\otimes^2 E_i\right)^*\right) \quad (m/2 \text{ factors})$$

and setting

$$\theta^{\otimes m/2}\left(f_1 \otimes \cdots \otimes f_{m/2}\right) = \theta(f_1) \otimes \cdots \otimes \theta(f_{m/2}), \ \text{ for } f_k \in (\otimes^2 E_i)^*.$$

Finally, define a homogeneous polynomial map of degree i:

$$S^m(E_i)^* \xhookrightarrow{\pi^*} (\otimes^m E_i)^* \overset{\theta^{\otimes m/2}}{\xrightarrow{\sim}} \otimes^{m/2}\left(M(i,i)\right) \xrightarrow{\det^{\otimes m/2}} \mathbb{C},$$

where the first map is induced from the canonical surjection $\pi : \otimes^m E_i \to S^m(E_i)$ and the last map $\det^{\otimes m/2}$ is given by

$$A_1 \otimes \cdots \otimes A_{m/2} \mapsto \det(A_1 \dots A_{m/2}),$$

for $A_k \in M(i,i)$.

Clearly, the composite map

$$\gamma_{m,i} : S^m(E_i)^* \to \mathbb{C},$$

where

$$\gamma_{m,i} := \det^{\otimes m/2} \circ \theta^{\otimes m/2} \circ \pi^*,$$

is a homogeneous polynomial of degree i, which is $SL(E_i)$-invariant.

Thus, we can think of $\gamma_{m,i} \in S^i(S^m(E_i))^{SL(E_i)}$.

Moreover, $\gamma_{m,i}$ is nonzero since $\gamma_{m,i}\left(\sum_{j=1}^i (e_j^*)^m\right) = 1$, where $\{e_1^*, \dots, e_i^*\}$ is the basis of E_i^* dual to the basis $\{e_i, \dots, e_i\}$ of E_i.

We record this in the following.

Lemma 2.6. *The element $\gamma_{m,i}$ is the unique (up to a scalar multiple) nonzero element of* $[S^i(S^m(E_i))]^{SL(E_i)}$.

3. Statement of the main theorem and its consequences

Now, let $\mathfrak{v}$ be a complex vector space of dimension m and let $E := \mathfrak{v}\otimes\mathfrak{v}^* = \operatorname{End}\mathfrak{v}$, $Q := S^m(E^*)$. Consider $\mathscr{D} \in Q$, where $\mathscr{D}$ is the function taking determinant of any $A \in E = \operatorname{End}\mathfrak{v}$. The group $G = GL(E)$ acts canonically on Q. Let $\mathcal{X}$ be the G-orbit closure of $\mathscr{D}$ inside Q.

Fix a basis $\{v_1, \dots, v_m\}$ of $\mathfrak{v}$ and let $\{v_1^*, \dots, v_m^*\}$ be the dual basis of $\mathfrak{v}^*$. Take the basis $\{v_i \otimes v_j^*\}_{1\le i,j\le m}$ of E and order them as $\{e_1, e_2, \dots, e_{m^2}\}$ satisfying

$$e_1 = v_1 \otimes v_1^*, \quad e_2 = v_2 \otimes v_2^*, \dots, e_m = v_m \otimes v_m^*.$$

Assume that m is even. Recall from Corollary 2.4 that for any $1 \le i \le m^2$, the irreducible $GL(E)$-module $V_E(m\omega_i)$ occurs in $S^i(S^m(E))$ with multiplicity one (and $V_E(m\omega_i)$ does not occur in any $S^j(S^m(E))$, for $j < i$). Let $P_i = \gamma_{m,i} \in S^i(S^m(E))$ be the highest weight vector of $V_E(m\omega_i) \subset S^i(S^m(E))$ (which is unique up to a nonzero scalar multiple) with respect to the standard Borel subgroup $B = B_E$ of G consisting of upper triangular invertible matrices, where $GL(E)$ is identified with $GL(m^2)$ with respect to the basis $\{e_1, \dots, e_{m^2}\}$ of E given above.

Recall an explicit construction of P_i from Lemma 2.6 (cf. Lemma 2.1). Since $P_i \in S^i(S^m(E))$, we can think of P_i as a homogeneous polynomial of degree i on the vector space $Q = S^m(E^*)$.

The following is our main result.

Theorem 3.1. *Assume, as above, that m is even. Then, with the above notation, for any $1 \le i \le m$, the polynomial P_i does* not *vanish identically on the orbit $GL(E) \cdot \mathscr{D}$.*

In particular, the irreducible $GL(E)$-module $V_E(m\omega_i)$ occurs with multiplicity one in the affine coordinate ring $\mathbb{C}[\mathcal{X}]$. Moreover, by Corollary 2.4, $V_E(d\omega_i)$, for any $d < m$ and any $1 \le i \le m^2$, does not occur in $S^\cdot(S^m(E))$; in particular, it does not occur in $\mathbb{C}[\mathcal{X}]$.

We postpone the proof of this theorem until the next section. But, we derive the following consequences.

Corollary 3.2. *With the notation and assumptions as in the last theorem, for any dominant integral weight λ for $GL(E)$ of the form $\lambda = \sum_{i=1}^m n_i\omega_i$, $n_i \in \mathbb{Z}_+$, the irreducible $GL(E)$-module $V_E(m\lambda)$ occurs in $\mathbb{C}[\mathcal{X}]$ with nonzero multiplicity.*

Proof. First of all, $\mathcal{X}$ being an irreducible variety, $\mathbb{C}[\mathcal{X}]$ is an integral domain. Take a B_E-eigenvector $\overline{P}_i \in \mathbb{C}[\mathcal{X}]$ of weight $m\omega_i$ for any $1 \le i \le m$;

which exists by the last theorem. Now, consider the function

$$\overline{P}_\lambda = \prod_{i=1}^{m} \overline{P}_i^{n_i} \in \mathbb{C}[\mathcal{X}].$$

Clearly, $\overline{P}_\lambda$ is a nonzero B_E-eigenvector of weight $m\lambda$. This proves the Corollary. ◻

Let $\mathcal{X}^o$ be the G-orbit $G \cdot \mathscr{D} \subset S^m(E^*)$. Then, by a classical result due to Frobenius (cf. [K, Proposition 2.1 and Corollary 2.3]), the isotropy subgroup $G_{\mathscr{D}}$ of $\mathscr{D}$ is a reductive subgroup. In particular, by a result of Matsushima, $\mathcal{X}^o$ is an affine variety. Moreover, by Frobenius reciprocity, we get the following.

Proposition 3.3. $\mathbb{C}[\mathcal{X}^o] \simeq \bigotimes_\lambda V_E(\lambda)\otimes[V_E(\lambda)^*]^{G_{\mathscr{D}}}$ *as G-modules, where the above summation runs over all the dominant integral weights λ of G (i.e., λ runs over $\sum_{i=1}^{m^2} n_i\omega_i$, $n_i \in \mathbb{Z}_+$ for all $1 \le i < m^2$ and $n_{m^2} \in \mathbb{Z}$) and $[V_E(\lambda)^*]^{G_{\mathscr{D}}}$ denotes the subspace of $G_{\mathscr{D}}$-invariants in the dual space $V_E(\lambda)^*$. The action of G on the right side is via its standard action on the first factor and it acts trivially on the second factor.*

In particular, the multiplicity of $V_E(\lambda)$ in $\mathbb{C}[\mathcal{X}^o]$ is the dimension of the invariant space $[V_E(\lambda)^*]^{G_{\mathscr{D}}}$.

Considering the action of the centre of G, it is easy to see that if $V_E(\lambda)$ occurs in $\mathbb{C}[\mathcal{X}^o]$, then $|\lambda| := \sum_{i=1}^{m^2} in_i \in m\mathbb{Z}$, where $\lambda = \sum_{i=1}^{m^2} n_i\omega_i$.

Again applying [K, Corollary 2.3] and [FH, Exercise 6.11(b), page 80], we get that for any polynomial representation $V_E(\lambda)$ (i.e., $\lambda = \sum_{i=1}^{m^2} n_i\omega_i$ with all $n_i \in \mathbb{Z}_+$) with $|\lambda| = md$, $d \in \mathbb{Z}_+$,

$$\dim\,[V_E(\lambda)^*]^{G_{\mathscr{D}}} \le k^{\overline{\lambda}}_{d\bar{\delta}_m, d\bar{\delta}_m}, \tag{1}$$

where $\bar{\delta}_m$ is the partition $\bar{\delta}_m : (1 \ge 1 \ge \cdots \ge 1)$ (m factors), $\overline{\lambda}$ is the partition $(n_1 + \cdots + n_{m^2} \ge n_2 + \cdots + n_{m^2} \ge n_3 + \cdots + n_{m^2} \ge \cdots \ge n_{m^2} \ge 0)$ and $k^{\overline{\lambda}}_{d\bar{\delta}_m, d\bar{\delta}_m}$ is the Kronecker coefficient (i.e., the multiplicity of the irreducible S_{dm}-module $L(\overline{\lambda})$ in the tensor product $L(d\bar{\delta}_m)\otimes L(d\bar{\delta}_m)$, where $L(\overline{\lambda})$ denotes the irreducible S_{dm}-module corresponding to the partition $\overline{\lambda}$).

As a corollary of the equation (1), and Proposition 3.3, we get the following (since $\mathbb{C}[\mathcal{X}] \hookrightarrow \mathbb{C}[\mathcal{X}^o]$ is a G-module embedding).

Corollary 3.4. *For any irreducible polynomial representation $V_E(\lambda)$ of G, such that $|\lambda| = dm$, for $d \in \mathbb{Z}_+$, the multiplicity $\mu(\lambda)$ of $V_E(\lambda)$ in $\mathbb{C}[\mathcal{X}]$ is bounded as below:*

$$\mu(\lambda) \le k^{\overline{\lambda}}_{d\bar{\delta}_m, d\bar{\delta}_m}.$$

Observe that unless $V_E(\lambda)$ is a polynomial representation of G and $|\lambda| \in m\mathbb{Z}_+$, $\mu(\lambda) = 0$.

As an immediate consequence of Corollaries 3.2 and 3.4, we get the following.

Corollary 3.5. *Let m be any positive even integer. Then, for any partition $\bar\lambda : \left(\bar\lambda_1 \geq \bar\lambda_2 \geq \cdots \geq \bar\lambda_m \geq 0\right)$ (with at most m parts) of d (i.e., $\sum_i \bar\lambda_i = d$), the Kronecker coefficient*

$$k^{m\bar\lambda}_{d\bar\delta_m, d\bar\delta_m} > 0.$$

In fact, by the above proof, as observed by Landsberg, we obtain that even the corresponding symmetric Kronecker coefficient is > 0.

Remark 3.6. Compare the above corollary with [BCI, Theorem 1, § 3].

4. Proof of Theorem 3.1

We continue to assume that m is even.

Define a right action of the semigroup $\mathrm{End}(E)$ on $Q = S^m(E^*)$ via

$$(f \odot A)(e) = f(Ae), \quad \text{for } f \in Q,\ A \in \mathrm{End}(E) \text{ and } e \in E.$$

To prove Theorem 3.1, it suffices to show that, for any $1 \leq i \leq m$, $P_i(\mathscr{D} \odot A) \neq 0$, for some $A \in \mathrm{End}(E)$.

We first consider an arbitrary $1 \leq i \leq m^2$.

In the following, we will restrict A to be of the form

$$Ae_j = \sum_{p=1}^{m} a^j_p e_p, \quad 1 \leq j \leq i, \tag{$*$}$$

where recall that, for any $1 \leq p \leq m$, $e_p = v_p \otimes v_p^* \in E$. (The values of Ae_j for $j > i$ will be irrelevant for us.)

The following is a crucial result.

Proposition 4.1. *For any A as above in $(*)$ and any $1 \leq i \leq m^2$, $P_i(\mathscr{D} \odot A)$ is the determinant of the $i \times i$ matrix $\mathscr{F}^i_A$ whose (j,k) term (for $1 \leq j,k \leq i$) $f_{j,k}$ is given as follows:*

$$f_{j,k} = \frac{(m/2-1)!}{m!} \sum_{\mathbf{d}} \frac{1}{\mathbf{d}!} \left(S_m \cdot a_{2\mathbf{d}+\delta_j+\delta_k}\right),$$

where the summation runs over $\mathbf{d} = (d_1, \ldots, d_i) \in \mathbb{Z}^i_+$ such that $|\mathbf{d}| := \sum_{j=1}^i d_j = m/2 - 1$.

Here δ_j denotes the i-tuple $\delta_j := (0, ..., 1, ..., 0)$, where 1 is placed in the j-th slot and 0 elsewhere; $2\mathbf{d}$ denotes the i-tuple $(2d_1, \ldots, 2d_i)$; for any i-tuple $\mathbf{t} = (t_1, \ldots, t_i) \in \mathbb{Z}^i_+$ with $|\mathbf{t}| = m$,

$$a_{\mathbf{t}} := \left(a^1_1 \cdots a^1_{t_1}\right)\left(a^2_{t_1+1} \cdots a^2_{t_1+t_2}\right) \cdots \left(a^i_{t_1+\cdots+t_{i-1}+1} \cdots a^i_{t_1+\cdots+t_i=m}\right); \tag{2}$$

for $\sigma \in S_m$,

(3) $\sigma \cdot a_{\mathbf{t}} := \left(a^1_{\sigma(1)} \cdots a^1_{\sigma(t_1)}\right)\left(a^2_{\sigma(t_1+1)} \cdots a^2_{\sigma(t_1+t_2)}\right) \cdots \left(a^i_{\sigma(t_1+\cdots+t_{i-1}+1)} \cdots a^i_{\sigma(m)}\right);$

$\mathbf{d}! := d_1! \cdots d_i!$, *and* $S_m \cdot a_{\mathbf{t}} = \sum_{\sigma \in S_m} \sigma \cdot a_{\mathbf{t}}$ *.*

Proof. First of all, by the explicit construction of $P_i = \gamma_{m,i}$ in § 2.5, $P_i(\mathscr{D} \odot A)$ is given by the determinant of the $i \times i$-matrix $\mathscr{F}^i_A$ whose (j,k) term $f_{j,k}$ $(1 \leq j,k \leq i)$ is given as follows:

$$f_{j,k} = \sum (\mathscr{D} \odot A)\left(e_j \cdot (e_{j_2})^2 \cdot (e_{j_3})^2 \cdots (e_{j_{m/2}})^2 \cdot e_k\right), \tag{4}$$

where the summation runs over the $(m/2-1)$-tuples $(j_2, j_3, \ldots, j_{m/2})$ with each $1 \leq j_p \leq i$, and the element $e_j \cdot e^2_{j_2} \cdots e^2_{j_{m/2}} \cdot e_k$ is considered as an element in $S^m(E)$.

The right side of the equation (4) can clearly be written as (with $m' := m/2 - 1$)

$$\begin{aligned} &\sum_{\substack{\mathbf{d}=(d_1,\ldots,d_i)\in\mathbb{Z}^i_+ \\ \text{with } |\mathbf{d}|=m'}} \binom{m'}{d_1}\binom{m'-d_1}{d_2}\binom{m'-d_1-d_2}{d_3}\cdots\binom{m'-(d_1+\cdots+d_{i-1})}{d_i} \\ &\qquad (\mathscr{D} \odot A)\left(e_j \cdot e_1^{2d_1} \cdot e_2^{2d_2} \cdots e_i^{2d_i} \cdot e_k\right) \\ &\qquad = \sum_{\substack{\mathbf{d}\in\mathbb{Z}^i_+: \\ |\mathbf{d}|=m'}} \frac{m'!}{\mathbf{d}!}(\mathscr{D} \odot A)\left(e_j \cdot e_1^{2d_1} \cdot e_2^{2d_2} \cdots e_i^{2d_i} \cdot e_k\right), \end{aligned} \tag{5}$$

where, as ealier, $\mathbf{d}! = d_1! \cdots d_i!$.

Now, for the indeterminates z_j, $1 \leq j \leq i$, observe that

$$\begin{aligned} \det(A \cdot (z_1e_1 + \cdots + z_ie_i)) &= (\mathscr{D} \odot A)((z_1e_1 + \cdots + z_ie_i)^m) \\ &= \sum_{\substack{\mathbf{t}=(t_1,\ldots,t_i)\in\mathbb{Z}^i_+ \\ \text{with } |\mathbf{t}|=m}} \frac{m!}{\mathbf{t}!}\mathbf{z}^{\mathbf{t}}(\mathscr{D} \odot A)\left(e_1^{t_1} \ldots e_i^{t_i}\right), \end{aligned} \tag{6}$$

where $\mathbf{z}^{\mathbf{t}} := z_1^{t_1} \ldots z_i^{t_i}$.

Denote the coefficient of $\mathbf{z}^{\mathbf{t}}$ in the above expression (6) by $(A \cdot (\mathbf{z} \cdot \mathbf{e}))_{\mathbf{t}}$. Thus, (6) can be written as

$$\det(A \cdot (z_1e_1 + \cdots + z_ie_i)) = \sum_{\mathbf{t}} (A \cdot (\mathbf{z} \cdot \mathbf{e}))_{\mathbf{t}}\mathbf{z}^{\mathbf{t}}, \tag{7}$$

where

$$(A \cdot (\mathbf{z} \cdot \mathbf{e}))_{\mathbf{t}} = \frac{m!}{\mathbf{t}!}(\mathscr{D} \odot A)\left(e_1^{t_1} \ldots e_i^{t_i}\right).$$

Combining this with the identities (4) and (5), we get

$$f_{j,k} = \sum_{\substack{\mathbf{d}\in\mathbb{Z}_+^i:\\ |\mathbf{d}|=m'}} \frac{m'!}{\mathbf{d}!}\frac{(2\mathbf{d}+\delta_j+\delta_k)!}{m!}(A\cdot(\mathbf{z}\cdot\mathbf{e}))_{2\mathbf{d}+\delta_j+\delta_k}. \tag{8}$$

Now,

$$\begin{aligned}
&\det(A\cdot(z_1e_1+\cdots+z_ie_i))\\
&=\det\left(\sum_{j=1}^{i} z_j\sum_{k=1}^{m} a_k^j e_k\right)\\
&=\prod_{k=1}^{m}\left(\sum_{j=1}^{i} z_j a_k^j\right)\\
&=\sum_{\substack{\mathbf{t}\in\mathbb{Z}_+^i:\\ |\mathbf{t}|=m}} \mathbf{z}^{\mathbf{t}} \sum_{\sigma\in S_m/S_{t_1}\times\cdots\times S_{t_i}} \left[\left(a^1_{\sigma(1)}\cdots a^1_{\sigma(t_1)}\right)\left(a^2_{\sigma(t_1+1)}\cdots a^2_{\sigma(t_1+t_2)}\right)\cdots\left(a^i_{\sigma(t_1+\cdots+t_{i-1}+1)}\cdots a^i_{\sigma(m)}\right)\right].
\end{aligned}$$

In particular, for any $\mathbf{d}\in\mathbb{Z}_+^i$: $|\mathbf{d}|=m'$ and $1\le j,k\le i$,

$$(A\cdot(\mathbf{z}\cdot\mathbf{e}))_{2\mathbf{d}+\delta_j+\delta_k} = \frac{1}{(2\mathbf{d}+\delta_j+\delta_k)!}\left(S_m\cdot a_{2\mathbf{d}+\delta_j+\delta_k}\right), \tag{9}$$

where $S_m.\ a_{2\mathbf{d}+\delta_j+\delta_k}$ is defined by the equation (3).

Thus, by (8) and (9), we get

$$f_{j,k} = \frac{\left(\frac{m}{2}-1\right)!}{m!}\sum_{\substack{\mathbf{d}\in\mathbb{Z}_+^i:\\ |\mathbf{d}|=m'}} \frac{1}{\mathbf{d}!}\left(S_m\cdot a_{2\mathbf{d}+\delta_j+\delta_k}\right). \tag{10}$$

This proves Proposition 4.1. □

Finally, we come to the proof of Theorem 3.1. We break the proof into two cases.

4.2. **Proof of Theorem 3.1.** By Proposition 4.1, it suffices to show that for any $1\le i\le m$, the determinant of the $i\times i$ matrix $\mathscr{F}_A^i$ is nonzero for some $A\in\mathrm{End}(E)$ of the form $(*)$ as in the beginning of this section, where $\mathscr{F}_A^i$ is the matrix $(f_{j,k})_{1\le j,k\le i}$ and $f_{j,k}$ is given by Proposition 4.1. We now deal with the two cases separately.

Case I, $1\le i\le \frac{m}{2}$: In this case choose $A\in\mathrm{End}(E)$ satisfying $(*)$ with the additional requirement as follows:

$$(C):\quad a_1^1, a_2^1\neq 0; a_3^2, a_4^2\neq 0;\ldots; a_{2i-3}^{i-1}, a_{2i-2}^{i-1}\neq 0; a_{2i-1}^i,\ldots,a_m^i\neq 0,$$

and all the rest $a_q^p=0$ for $1\le p\le i$, $1\le q\le m$.

In this case it is easy to see that $f_{j,k} = 0$, unless $j = k$. Moreover,

$$f_{j,j} = \frac{\left(\frac{m}{2}-1\right)!}{m!} \sum_{\substack{\mathbf{d}\in\mathbb{Z}_+^i:\\ |\mathbf{d}|=m'}} \frac{1}{\mathbf{d}!}\left(S_m \cdot a_{2\mathbf{d}+2\delta_j}\right).$$

Now, for $1 \leq j < i$, $S_m \cdot a_{2\mathbf{d}+2\delta_j} = 0$ unless

$$\mathbf{d} = \left(1, \dots, 1, 0, 1, \dots, 1, \frac{m}{2} + 1 - i\right),$$

where 0 is placed in the j-th slot. Thus,

$$f_{j,j} = \frac{\left(\frac{m}{2}-1\right)!(m-2i+2)!2^{i-1}}{m!\left(\frac{m}{2}+1-i\right)!}\alpha, \quad \text{for} \quad j < i, \tag{11}$$

where

$$\alpha := \left(a_1^1 a_2^1\right)\left(a_3^2 a_4^2\right)\dots\left(a_{2i-3}^{i-1} a_{2i-2}^{i-1}\right)\left(a_{2i-1}^i a_{2i}^i \dots a_m^i\right). \tag{12}$$

Similarly, for $j = i$, $S_m \cdot a_{2\mathbf{d}+2\delta_i=0}$ unless $\mathbf{d} = \left(1, \dots, 1, \frac{m}{2} - i\right)$.
Thus,

$$f_{i,i} = \frac{\left(\frac{m}{2}-1\right)!}{m!} \frac{(m-2i+2)!2^{i-1}}{\left(\frac{m}{2}-i\right)!}\alpha. \tag{13}$$

Combining (11) and (13) we get, for the choice of A given by (C),

$$\det(\mathscr{F}_A^i) = \left(\frac{\left(\frac{m}{2}-1\right)!(m-2i+2)!2^{i-1}}{m!\left(\frac{m}{2}-i+1\right)!}\alpha\right)^i \left(\frac{m}{2} - i + 1\right).$$

In particular, $\det(\mathscr{F}_A^i) \neq 0$ in this case, thus Theorem 3.1 is proved in the case $1 \leq i \leq m/2$.

Case II, $m/2 < i \leq m$: Let $i' := i - m/2$. In this case choose $A \in \mathrm{End}(E)$ satisfying ($*$) with the additional requirement as follows:

$$a_1^1,\ a_2^1 \neq 0;\ a_3^2,\ a_4^2 \neq 0; \dots; a_{m-1}^{m/2},\ a_m^{m/2} \neq 0;$$

$$a_1^{m/2+1},\ a_2^{m/2+1} \neq 0; \dots;\ a_{2i'-1}^i,\ a_{2i'}^i \neq 0,$$

and all the rest $a_q^p = 0$ for $1 \leq p \leq i$, $1 \leq q \leq m$. With such a choice of $A \in \mathrm{End}(E)$, we consider

$$P_i(\mathscr{D} \odot A) = \text{ determinant of } \mathscr{F}_A^i = \det(f_{j,k})_{1\leq j,k\leq i}$$

as a polynomial in the above variables

$$\left\{a_{2j-1}^j,\ a_{2j}^j\right\}_{1\leq j\leq m/2} \cup \left\{a_{2j-1}^{m/2+j}, a_{2j}^{m/2+j}\right\}_{1\leq j\leq i'}.$$

We now calculate the entries $f_{j,k}$.

Observe that $f_{j,k} = 0$ for all the pairs (j, k) except the following three types of pairs:

(p_1) $1 \leq j = k \leq i$
(p_2) $k = j + m/2$ and $1 \leq j \leq i'$
and
(p_3) $j = k + m/2$ and $1 \leq k \leq i'$.

To prove this, observe that for any pair $1 \leq j, k \leq i$ which is not of any of the above three types, any translate $\sigma \cdot a_{2\mathbf{d}+\delta_j+\delta_k}$, for any $\mathbf{d} \in \mathbb{Z}^i_+$ with $|\mathbf{d}| = m'$, has at most $m-2$ nonzero components; in particular, it has at least 2 components which are zero. Thus, $\sigma \cdot a_{2\mathbf{d}+\delta_j+\delta_k} = 0$, for all $\sigma \in S_m$ and hence $S_m \cdot a_{2\mathbf{d}+\delta_j+\delta_k} = 0$.

Thus, the $i \times i$ matrix $\mathscr{F}^i_A$ being symmetric is given by:

$$\begin{array}{cc} & \begin{array}{ccc} \qquad i' \qquad & \qquad m/2 - i' \qquad & \qquad i' \qquad \end{array} \\ \begin{array}{c} i' \\[4em] m/2 - i' \\[4em] i' \end{array} & \left(\begin{array}{c|c|c} \begin{matrix} f_{1,1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & f_{i',i'} \end{matrix} & \bigcirc & \begin{matrix} f_{1,1+m/2} & & \bigcirc \\ & \ddots & \\ \bigcirc & & f_{i',i'+m/2} \end{matrix} \\ \hline \bigcirc & \begin{matrix} f_{i'+1,i'+1} & & \bigcirc \\ & \ddots & \\ \bigcirc & & f_{m/2,m/2} \end{matrix} & \bigcirc \\ \hline \begin{matrix} f_{1,1+m/2} & & \bigcirc \\ & \ddots & \\ \bigcirc & & f_{i',i'+m/2} \end{matrix} & \bigcirc & \begin{matrix} f_{1+m/2,1+m/2} & & \bigcirc \\ & \ddots & \\ \bigcirc & & f_{i,i} \end{matrix} \end{array}\right) \end{array}$$

Its determinant can easily be calculated, which is

$$P_i(\mathscr{D} \odot A) = (f_{i'+1,i'+1} \cdots f_{m/2,m/2}) \prod_{j=1}^{i'} \left(f_{j,j} f_{m/2+j,m/2+j} - f^2_{j,j+m/2} \right). \tag{14}$$

We now calculate $f_{j,k}$ for the above pairs (p_1)-(p_3). Since the matrix $(f_{j,k})$ is symmetric, we only need to calculate $f_{j,k}$ for the first two types of pairs (p_1)-(p_2).

We first define a grading in the polynomial ring

$$\mathbb{C}\left[a^j_{2j-1}, a^j_{2j}, a^{m/2+t}_{2t-1}, a^{m/2+t}_{2t}\right]_{\substack{1 \leq j \leq m/2, \\ 1 \leq t \leq i'}}$$

by setting

$$\text{degree } a^j_{2j-1} = \text{ degree } a^j_{2j} = 2, \quad \text{for all} \quad 1 \le j \le m/2$$

and

$$\text{degree } a^{m/2+t}_{2t-1} = \text{ degree } a^{m/2+t}_{2t} = 1, \quad \text{for all} \quad 1 \le t \le i'.$$

We now calculate the top degree homogeneous component $f^o_{j,k}$ of $f_{j,k}$ for the pairs (p_1)-(p_2).

From Proposition 4.1 for the expression of $f_{j,k}$, we easily see that

$$f^o_{j,j} = \mu, \quad \text{for} \quad 1 \le j \le m/2, \tag{15}$$

where

$$\mu := \frac{(m/2-1)!}{m!} 2^{m/2} \left(a^1_1 a^1_2\right)\left(a^2_3 a^2_4\right)\dots\left(a^{m/2}_{m-1} a^{m/2}_m\right).$$

Similarly,

$$f^o_{j,j} = \frac{\mu a^j_{2j-m-1} a^j_{2j-m}}{a^{j-m/2}_{2j-m-1} a^{j-m/2}_{2j-m}}, \quad \text{for} \quad m/2 < j \le i. \tag{16}$$

Finally,

$$f^o_{j,j+m/2} = \frac{\mu}{2} \frac{a^j_{2j-1} a^{j+m/2}_{2j} + a^j_{2j} a^{j+m/2}_{2j-1}}{a^j_{2j-1} a^j_{2j}}, \quad \text{for} \quad 1 \le j \le i'. \tag{17}$$

Thus, from the identities (14) - (17), the top degree homogeneous component $P^o_i(\mathscr{D} \odot A)$ of $P_i(\mathscr{D} \odot A)$ is given as follows:

$$P^o_i(\mathscr{D} \odot A) = \mu^{m/2-i'} \prod_{j=1}^{i'} \left(\mu f^o_{j+m/2,j+m/2} - (f^o_{j,j+m/2})^2\right). \tag{18}$$

But, for any $1 \le j \le i'$, by the equations (16), (17),

$$\begin{aligned}
\mu f^o_{j+m/2,j+m/2} - \left(f^o_{j,j+m/2}\right)^2 \\
&= \mu^2 \frac{a^{j+m/2}_{2j-1} a^{j+m/2}_{2j}}{a^j_{2j-1} a^j_{2j}} - \frac{\mu^2}{4}\left(\frac{a^j_{2j-1} a^{j+m/2}_{2j} + a^j_{2j} a^{j+m/2}_{2j-1}}{a^j_{2j-1} a^j_{2j}}\right)^2 \\
&= -\left(\frac{\mu}{2a^j_{2j-1} a^j_{2j}}\right)^2 \left(a^j_{2j-1} a^{j+m/2}_{2j} - a^j_{2j} a^{j+m/2}_{2j-1}\right)^2.
\end{aligned}$$

This is clearly a nonzero homogeneous polynomial. Thus, by the identity (18), $P^o_i(\mathscr{D} \odot A)$ is a nonzero polynomial and hence so is $P_i(\mathscr{D} \odot A)$. This completes the proof of Theorem 3.1 in the case $m \ge i > m/2$ and hence the theorem is completely established. □

5. Some additional results

Let m be even. By virtue of Theorem (3.1), for any $1 \leq i \leq m$, the irreducible $GL(E)$-module $V_E(m\omega_i)$ occurs with multiplicity one in the affine coordinate ring $\mathbb{C}[\mathcal{X}]$ and $V_E(d\omega_i)$ for any $1 \leq d < m$ and $1 \leq i \leq m^2$ does not occur in $\mathbb{C}[\mathcal{X}]$. I do not know if $V_E(m\omega_i)$ occurs in $\mathbb{C}[\mathcal{X}]$, for $m < i < m^2$. However, we have the following 'negative' result.

Proposition 5.1. *Let the notation and assumptions be as in Theorem 3.1. In particular, m is even. Assume further that $m > 2$. Then, the polynomial P_{m^2} vanishes identically on the orbit $GL(E) \cdot \mathscr{D}$.*

Thus, the irreducible G-module $V_E(m\omega_{m^2})$ does not occur in the affine coordinate ring $\mathbb{C}[\mathcal{X}]$.

Proof. The highest weight vector $P_{m^2} \in S^{m^2}(S^m(E))$ (of highest weight $m\omega_{m^2}$) is in fact $SL(E)$-invariant. Of course, the center of $GL(E)$ acts via scalar multiplication on $S^{m^2}(S^m(E))$; in particular, on P_{m^2}. Hence, it suffices to prove that

$$P_{m^2}(\mathscr{D}) = 0. \tag{19}$$

By the identity (4) of Section 4, $P_{m^2}(\mathscr{D})$ is given by the determinant of the $m^2 \times m^2$ matrix $\mathscr{F}_I^{m^2}$ whose (j,k) term $(1 \leq j,k \leq m^2)$ is given by

$$f_{j,k} = \sum \mathscr{D}\left(e_j \cdot (e_{j_2})^2 \cdots (e_{j_{m/2}})^2 \cdot e_k\right), \tag{20}$$

where the summation runs over the $(m/2-1)$-tuples $(j_2, \dots, j_{m/2})$ with each $1 \leq j_p \leq m^2$ and $\{e_j\}$ is the basis of E as in the beginning of Section 3.

Now, since $x = e_j \cdot (e_{j_2})^2 \cdots (e_{j_{m/2}})^2 \cdot e_k$ is an element of $S^m(E)$, we can write

$$x = \sum_p v_p^m, \quad \text{for } v_p \in \mathbb{C}e_j \oplus \mathbb{C}e_{j_2} \oplus \cdots \oplus \mathbb{C}e_{j_{m/2}} \oplus \mathbb{C}e_k \subset E.$$

Thus,

$$\begin{aligned} \mathscr{D}\left(e_j \cdot (e_{j_2})^2 \cdots (e_{j_{m/2}})^2 \cdot e_k\right) &= \sum_p \mathscr{D}(v_p^m) \\ &= \sum_p \det(v_p) \\ &= 0. \end{aligned} \tag{21}$$

The vanishing of $\det(v_p)$ follows since $v_p \in E$ lies in the span of $m/2+1$ basis vectors $e_j's$; in particular, as a $m \times m$ matrix, v_p has at most $m/2+1$ nonzero entries. Thus, for $m > 2$, $\det(v_p) = 0$ and hence by the identities (20)-(21), $f_{j,k} = 0$ for all $1 \leq j, k \leq m^2$. In particular, $P_{m^2}(\mathscr{D}) = \det(f_{j,k}) = 0$. This proves the proposition. □

Remark 5.2. The above proposition is indeed false for $m = 2$, since in this case $\mathcal{X} = S^2(E^*)$ (cf. [K, Example 2.7]) and hence P_4 does not vanish on $\mathcal{X}$.

For any $1 \leq i \leq m^2$, and any $A \in \operatorname{End}(E)$ of the form $(*)$ (as in the beginning of Section 4), we give another expression for $P_i(\mathscr{D} \odot A) = \det \mathscr{F}_A^i$.

Proposition 5.3. *With the notation as above, setting $m' = m/2 - 1$,*

$$P_i(\mathscr{D} \odot A) = \left(\frac{m'!}{m!}\right)^i \sum_{1 \leq p_1, \dots, p_i \leq m} \left[((1, p_1)S_{m-1}) \cdot \left(a_2^1 \sum_{\substack{\mathbf{d} \in Z_+^i: \\ |\mathbf{d}| = m'}} \frac{a_{2\mathbf{d}}}{\mathbf{d}!}\right)\right] \cdots$$

$$\left[((1, p_i)S_{m-1}) \cdot \left(a_2^i \sum_{\substack{\mathbf{d} \in Z_+^i: \\ |\mathbf{d}| = m'}} \frac{a_{2\mathbf{d}}}{\mathbf{d}!}\right)\right] \begin{vmatrix} a_{p_1}^1 & \dots & a_{p_i}^1 \\ \vdots & & \vdots \\ a_{p_1}^i & \dots & a_{p_i}^i \end{vmatrix},$$

where, for $\mathbf{t} \in Z_+^i$: $|\mathbf{t}| = 2m'$,

$$a_{\mathbf{t}} := \left(a_3^1 a_4^1 \cdots a_{2+t_1}^1\right)\left(a_{3+t_1}^2 \cdots a_{2+t_1+t_2}^2\right) \cdots \left(a_{3+t_1+\cdots+t_{i-1}}^i \cdots a_{2+t_1+\cdots+t_i}^i\right),$$

$\sigma \cdot a_{\mathbf{t}}$ is defined by equation (3) *of § 4, S_{m-1} is the subgroup of S_m taking $1 \mapsto 1$, $((1, p_j)S_{m-1}) \cdot a_{\mathbf{t}}$ stands for $\sum_{\sigma \in (1, p_j)S_{m-1}} \sigma \cdot a_{\mathbf{t}}$ and $(1, p_j)$ is the transposition taking 1 to p_j.*

Proof. By Proposition 4.1, the k-th column C^k of the matrix $\mathscr{F}_A^i$ is given by

$$C^k = \begin{pmatrix} \frac{m'!}{m!} \sum_{1 \leq p \leq m} a_p^1((1, p)S_{m-1}) \cdot \left(a_2^k \sum_{\substack{\mathbf{d} \in \mathbb{Z}_+^i: \\ |\mathbf{d}| = m'}} \frac{a_{2\mathbf{d}}}{\mathbf{d}!}\right) \\ \vdots \\ \frac{m'!}{m!} \sum_{1 \leq p \leq m} a_p^i((1, p)S_{m-1}) \cdot \left(a_2^k \sum_{\substack{\mathbf{d} \in Z_+^i: \\ |\mathbf{d}| = m'}} \frac{a_{2\mathbf{d}}}{\mathbf{d}!}\right) \end{pmatrix}$$

$$= \frac{m'!}{m!} \sum_{p=1}^m ((1, p)S_{m-1}) \cdot \left(a_2^k \sum_{\substack{\mathbf{d} \in \mathbb{Z}_+^i: \\ |\mathbf{d}| = m'}} \frac{a_{2\mathbf{d}}}{\mathbf{d}!}\right) \begin{pmatrix} a_p^1 \\ \vdots \\ a_p^i \end{pmatrix}.$$

From this the proposition follows easily. □

Corollary 5.4. *For any even positive integer m, any $i > m$, and any $A \in$* $\operatorname{End}(E)$ *of the form* $(*)$ *(as in the beginning of Section 4),*

$$P_i(\mathscr{D} \odot A) = 0.$$

Proof. In the expression of $P_i(\mathscr{D} \odot A)$ given by the above proposition, the determinant

$$\begin{vmatrix} a^1_{p_1} & \dots & a^1_{p_i} \\ \vdots & & \vdots \\ a^i_{p_1} & \dots & a^i_{p_i} \end{vmatrix}$$

is clearly zero unless $p_j \neq p_k$ for all $1 \leq j \neq k \leq i$. This is of course only possible if $i \leq m$. This proves the corollary. □

Remark 5.5. (a) Observe that the above corollary does not imply that for $i > m$, P_i vanishes identically on $\mathcal{X}$. In the above corollary, we are only taking A of the special form $(*)$ which is not a general element of $\mathrm{End}(E)$. I do not know if P_i for $2m < i < m^2$ vanishes identically on $\mathcal{X}$. Of course, by Theorem 3.1, P_i does not vanish identically on $\mathcal{X}$ for $1 \leq i \leq m$; by the following remark (d), P_i does not vanish identically on $\mathcal{X}$ for $m+1 \leq i \leq 2m$; and, by Proposition 5.1, P_{m^2} vanishes identically on $\mathcal{X}$.

(b) By exactly the same proof as given in Section 4, for any positive even integer m and any $1 \leq i \leq m$, the polynomial P_i does not vanish identically on the orbit $GL(E) \cdot \mathcal{P}$, where $\mathcal{P}$ is the function $E \to \mathbb{C}$ taking any matrix $A \in E := \mathrm{End}\, \mathfrak{v}$ to its permanent. In fact, the analogue of the matrix $\mathscr{F}^i_A$ (for any A satisfying $(*)$ as in the beginning of Section 4) for $\mathscr{D} \odot A$ replaced by $\mathcal{P} \odot A$ is identical to the matrix $\mathscr{F}^i_A$ (since the determinant and permanent of a diagonal matrix are the same).

In particular, the irreducible $GL(E)$-module $V_E(m\omega_i)$ occurs with multiplicity one in $\mathbb{C}[\overline{GL(E) \cdot \mathcal{P}}]$ for any $i \leq m$. Moreover, $V_E(d\omega_i)$, for any $d < m$ and $1 \leq i \leq m^2$ does not occur in $\mathbb{C}[\overline{GL(E) \cdot \mathcal{P}}]$ (cf. Corollary 2.4).

(c) Even though we do not have any application in mind, the following generalization of the above remark (b) holds by exactly the same proof as given in Section 4.

Let $\mathscr{F} \in Q := S^m(E^*)$ be any (homogeneous) polynomial such that writing $\mathscr{F}$ as a sum of monomials (in a basis of E^*), some monomial with no repeated factors occurs with nonzero coefficient. Then, for any positive even integer m and any $1 \leq i \leq m$, the polynomial P_i does not vanish identically on the orbit $GL(E) \cdot \mathscr{F}$.

(d) (Due to J. Landsberg) Let C be the Chow subvariety of Q, which is, by definition, the $GL(E)$-orbit closure of the monomial $e^*_1 \cdots e^*_m$ (for any basis $\{e^*_1, \dots, e^*_m, \dots, e^*_{m^2}\}$ of E^*). Further, let $\mathcal{S}$ be the second secant variety of C, which is, by definition, the $GL(E)$-orbit closure of $e^*_1 \cdots e^*_m + e^*_{m+1} \cdots e^*_{2m}$. Then, suitably adapting the proof of Theorem 3.1, and using the fact that $\mathcal{S} \subset \mathcal{X}$, Landsberg has shown that any P_i (for $m + 1 \leq i \leq 2m$) does not vanish identically on $\mathcal{S}$ (and hence on $\mathcal{X}$). In particular, the irreducible

$GL(E)$-module $V_E(m\omega_i)$ occurs with multiplicity one in $\mathbb{C}[\mathcal{X}]$ for any $1 \leq i \leq 2m$.

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Address: Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA
(email: shrawan@email.unc.edu)